Before we talk about energy...

There is an application of Calculus that you will do a great deal of in first year university called TAYLOR EXPANSION. This application allows you to express (approximate) any function with an infinitely long polynomial.

It works like this: $$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\left[\frac{d}{dx}\right]^n f(x)\right)\Bigg|_{x=\emptyset}$$

0TH, 1ST, 2ND, 3RD Derivative etc.

factorial

EVALUATED AT $x = \emptyset$

So the first term in your Polynomial is when $n=\emptyset$ $\longrightarrow$ $\frac{x^0}{0!}\left(\left[\frac{d}{dx}\right]^0 f(x)\right)\Big|_{x=0} = f(0)$ easy (constant)

the Second: $n=1 \rightarrow \frac{x^1}{1!}\left(\left[\frac{d}{dx}\right]^1 f(x)\right)\Big|_{x=0} = \left(\frac{df}{dx}\right)\Big|_{x=0} x$ (linear)

the Third: $n=2 \longrightarrow \frac{x^2}{2!}\left(\left[\frac{d}{dx}\right]^2 f(x)\right)\Big|_{x=0} = \frac{1}{2}\left(\frac{d^2f}{dx^2}\right)\Big|_{x=0} x^2$ (quadratic)

the Fourth: $n=3 \longrightarrow \frac{x^3}{3!}\left(\left[\frac{d}{dx}\right]^3 f(x)\right)\Big|_{x=0} = \frac{1}{6}\left(\frac{d^3f}{dx^3}\right)_{x=0} x^3$ (cubic)

the Fifth: $n=4 \longrightarrow \frac{x^4}{4!}\left(\left[\frac{d}{dx}\right]^4 f(x)\right)\Big|_{x=0} = \frac{1}{24}\left(\frac{d^4f}{dx^4}\right)_{x=0} x^4$ (quartic)

Try it with $f(x) = 4x^3 - 5x^2 + 3x + 4 \rightarrow f(0) = 4$ etc.

$\frac{df}{dx} = 12x^2 - 10x + 3 \longrightarrow \frac{df}{dx}\Big|_{x=0} = 3$

$\frac{d^2f}{dx^2} = 24x - 10 \longrightarrow \frac{d^2f}{dx^2}\Big|_{x=0} = -10$

$\frac{d^3f}{dx^3} = 24 \longrightarrow \frac{d^3f}{dx^3}\Big|_{x=0} = 24$

$\frac{d^4f}{dx^4} = \emptyset \longrightarrow \frac{d^4f}{dx^4}\Big|_{x=0} = \emptyset$ as will every term after.

because we used a polynomial, the sequence will terminate:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\left[\frac{d}{dx}\right]^n f(x)\right)\Bigg|_{x=0} = f(0) + \left(\frac{df}{dx}\right)_{x=0} x + \frac{1}{2}\left(\frac{d^2f}{dx^2}\right)_{x=0} x^2 + \frac{1}{6}\left(\frac{d^3f}{dx^3}\right)_{x=0} x^3 + \frac{1}{24}\left(\frac{d^4f}{dx^4}\right)_{x=0} x^4$$

$$= (4) + (3)x + \frac{1}{2}(-10)x^2 + \frac{1}{6}(24)x^3 + \frac{1}{24}(24)x^4$$

$$= 4 + 3x - 5x^2 + 4x^3 + 1x^4$$ which is the polynomial we started with... (NEAT ☺)

So now that you know it works, I am going to do the Taylor Expansion of γ, so I need its derivatives (v is the variable)

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$ $\therefore \gamma(0) = 1$

$$\frac{d\gamma}{dv} = -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2v}{c^2}\right) = \frac{1}{c^2} v \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$ $\therefore \frac{d\gamma(0)}{dv} = \varnothing$

$$\frac{d^2\gamma}{dv^2} = \frac{1}{c^2}\left\{[1]\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} + v\left[\frac{-3}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}}\left(\frac{-2v}{c^2}\right)\right]\right\}$$

$$\frac{d^2\gamma}{dv^2} = \frac{1}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}}\left(1 + \frac{2v^2}{c^2}\right)$$ $\therefore \frac{d^2\gamma(0)}{dv^2} = \frac{1}{c^2}$

$$\frac{d^3\gamma}{dv^3} = \frac{1}{c^2}\left\{\left[\frac{-5}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{7}{2}}\left(\frac{-2v}{c^2}\right)\right]\left(1 + \frac{2v^2}{c^2}\right) + \left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}}\left[\frac{4v}{c^2}\right]\right\}$$

$$\frac{d^3\gamma}{dv^3} = \frac{1}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{7}{2}}\left(\frac{9v}{c^2} + \frac{6v^3}{c^4}\right)$$ $\therefore \frac{d^3\gamma(0)}{dv^3} = \varnothing$

$$\frac{d^4\gamma}{dv^4} = \frac{1}{c^2}\left\{\left[\frac{-7}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{9}{2}}\left(\frac{-2v}{c^2}\right)\right]\left(\frac{9v}{c^2} + \frac{6v^3}{c^4}\right) + \left(1 - \frac{v^2}{c^2}\right)^{-\frac{7}{2}}\left[\frac{9}{c^2} + \frac{18v^2}{c^4}\right]\right\}$$

$$\frac{d^4\gamma}{dv^4} = \frac{1}{c^4}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{9}{2}}\left(\frac{24v^4}{c^4} + \frac{72v^2}{c^2} + 9\right)$$ $\therefore \frac{d^4\gamma(0)}{dv^4} = \frac{9}{c^4}$

And so on...

Hilroy

$$\therefore \gamma = \gamma(0) + \frac{d\gamma(0)}{dv} v + \frac{1}{2}\frac{d^2\gamma(0)}{dv^2} v^2 + \frac{1}{6}\frac{d^3\gamma(0)}{dv^3} v^3 + \frac{1}{24}\frac{d^4\gamma(0)}{dv^4} v^4 \cdots \text{etc.}$$

fill in the numbers from the last page:

$$\gamma = (1) + (\varnothing) v + \frac{1}{2}\left(\frac{1}{c^2}\right) v^2 + \frac{1}{6}(\varnothing) v^3 + \frac{1}{\underset{8}{\cancel{24}}}\left(\frac{\overset{3}{\cancel{9}}}{c^4}\right) v^4 \cdots \text{etc}$$

$$\gamma = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} \cdots \text{etc} \ldots$$

Remember this for later...

Thinking back to the Spacetime Interval

some purely spatial part

$$\vec{S} = (x, y, z, ct) \quad \text{OR} \quad \vec{S} = (\vec{d}, ct)$$

and since "Chuck" always saw himself as "proper" regardless of how silly Edwina's observations of him were, we will refer to Chuck's time as "proper time", and we will give it the symbol τ, while the observer's time will become t.

chain Rule

$\therefore$ "Proper velocity" should be $\dfrac{d\vec{S}}{d\tau} = \left(\dfrac{d\vec{d}}{d\tau}, \dfrac{d(ct)}{d\tau}\right) = \left(\dfrac{dt}{d\tau}\dfrac{d\vec{d}}{dt}, c\dfrac{dt}{d\tau}\right)$

but $\dfrac{dt}{d\tau} = \gamma$ (remember?) $\quad \therefore \dfrac{d\vec{S}}{d\tau} = \left(\gamma \dfrac{d\vec{d}}{dt}, c\gamma\right)$

The velocity Edwina observes the train having ($\vec{v}$)

Remember not to ask why it's negative

$$\therefore \left|\frac{d\vec{S}}{d\tau}\right|^2 = \left(\gamma^2|\vec{v}|^2 - \gamma^2 c^2\right) = \gamma^2\left(|\vec{v}|^2 - c^2\right) = \left(\frac{1}{\cancel{1-\frac{v^2}{c^2}}}\right)\left(\cancel{-\frac{|v|^2}{c^2}+1}\right)\left(-c^2\right) = -c^2$$

So, no matter how quickly Chuck is moving through SPACE his proper velocity across the SPACETIME INTERVAL is $-c^2$ (a constant)

Weird...

If we go back to momentum for a moment: $\vec{p} = m\vec{v}$

PROPER MOMENTUM $|\vec{p}|^2 = M^2\left|\frac{d\vec{s}}{d\tau}\right|^2 = -M^2c^2$ ← regardless of his speed through SPACE his proper momentum across the SPACETIME INTERVAL is $-mc^2$ (a constant)

PROPER MASS

OBSERVED MOMENTUM $|\vec{p}|^2 = m^2\left|\frac{d\vec{s}}{dt}\right|^2 = \gamma^2 M^2\left|\left(\frac{d}{dt}\vec{d}, \frac{d}{dt}(ct)\right)\right| = \gamma^2 M^2 \left|\vec{v}, c\right|$

OBSERVED MASS

$(m = \gamma M)$

The spatial part ($|\vec{v}|$)

The temporal part

∴ OBSERVED MOMENTUM $|\vec{p}|^2 = \gamma^2 M^2\left(|\vec{v}|^2 - c^2\right) = (\gamma M|\vec{v}|)^2 - (\gamma Mc)^2$

but Proper Momentum must match observed momentum.

$$\therefore -(Mc)^2 = (\gamma M|\vec{v}|)^2 - (\gamma Mc)^2$$

$$(\gamma Mc)^2 - (Mc)^2 = (\gamma M|\vec{v}|)^2$$

This is the ordinary "observed momentum" through space,

what is this, then?

Look at just $Mc\gamma$ and recall that the Taylor Expansion of $\gamma = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} \dots$

$$\therefore Mc\gamma = Mc + \frac{Mv^2}{2c} + \frac{3Mv^4}{8c^3} \dots$$

$$\therefore Mc^2\gamma = Mc^2 + \frac{1}{2}Mv^2 + \frac{3}{8}\frac{Mv^4}{c^2} \dots$$

$\gamma M = m$

KINETIC ENERGY!! ∴ Each of these terms must be an energy and $Mc^2\gamma = mc^2$ must be the total energy...

The total relativistic energy

$$mc^2 = Mc^2 + \frac{1}{2}Mv^2 + \frac{3}{8}\frac{Mv^4}{c^2} \text{ etc...}$$

but something odd happens when the observed object stops moving:

for $v = \emptyset$, $\underbrace{\text{total relativistic energy}}_{E} = Mc^2 + \frac{1}{2}M(0)^2 + \frac{3}{8}\frac{M(0)^4}{c^2} \cdots$ etc

$$E = Mc^2$$

* Even if you are not travelling through space ($v = \emptyset$) you are still travelling through time, and any object that is passing through time must be carrying an amount of energy proportional to their proper mass.

* Since objects CANNOT SIMPLY STOP PASSING THROUGH TIME, THE PRESENCE of MASS DEMANDS the presence of energy, which implies that energy + mass are the same thing...

Something to consider:

50g of Roasted Almonds claims to provide 320 calories ~ 1400J.
(that's what your body can extract from them chemically)

But what if you could UNDO them, and release all the matter as pure energy?

The almonds are at rest, so their Proper Mass is 50g = 0.05 kg.

So if $E = Mc^2 = (0.05)(3\times10^8)^2$
$= (0.05)(9\times10^{16})$
$= 4.5\times10^{15}$ J

$= 4.5$ PetaJoules ☺

10^3 kilo
10^6 Mega
10^9 Giga
10^{12} Tera
10^{15} Peta

Your digestive tract takes 24 hours to do its job. If you spread this energy over that time:

Power $= \frac{4.5\times10^{15}}{24\times60\times60} = 52.08$ TW this is 7000 times the output of Bruce Nuclear. ☺

Hilroy

A common trope in science fiction is the "transporter" which converts people directly to energy, and then "beams" them somewhere else in a matter of seconds...

Just how much energy is tied up in an 80kg person?

Even if the "beaming" process took a full minute, what would the power of the output beam be?

The Bruce Nuclear facility is the largest functioning nuclear power plant on Earth (Fukushima was the only one larger). It produces 7.3 Gigawatts of power (that is, 7.3×10^9 Joules per second)

How much mass needs to be converted to energy each second in order to produce this amount of energy?

The wind farm in Melancthon has 133 turbines with a maximum output of 1.5MW each. How many of these wind farms are needed to replace Bruce Nuclear?

The power output of the Sun is 3.846×10^{26} W. How much matter is the sun shedding as energy every second?